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Some properties of oscillation solutions of Sturm-Liouville equation with transmission conditions

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Abstract :
Oscillation theory for the solutions of Sturm-Liouville problems is one of the traditional trends in the qualitative theory of differential equations. Its main goal is to establish sufficient conditions for the existence of oscillating solutions, to investigate the laws of distribution of the zeros, the maxima and minima of the solution, to find estimates of the distance between the consecutive zeros and of the number of zeros in a given interval, as well as to obtain the relationship between the oscillatory and other fundamental properties of the solutions of various classes of differential equations. It is well-known that Sturm-Liouville type differential equations with classical boundary conditions arise after on application of the method of separation of variables to the varied assortment of physical problems. Recently such type boundary value problems under additional transmission conditions are investigated by many researches. In this paper, we investigate analogues of the classical Sturm comparison and oscillation theorems for discontinuous Sturm-Liouville problem together with transmission conditions. We present a new criteria for Sturm's comparison and oscillation theorems, discuss the main tools used in deriving those criteria.

Keywords : Sturm-liouville Problem, Sturm's comparison theorem, transmission conditions, oscillation.

1 Introduction

The significant second-order equation $y''+qy=0$ is important a valid subject for research, although it has a voluminous literature. Sturm-Liouville comparison and oscillation theory can be applied to establish some qualitative properties of the solutions to some of the various type differential equations that we cannot solve explicitly.

After Sturm's familiar work [1] in 1836, Sturmian comparison theorems have been derived for differential equations of various types. In order to obtain Sturmian comparison theorems for more general differential equations of second order, Picone [2] established an valuable identity, known as the Picone identity. In the latter years, Jaro. and Kusano

[3, 5] derived a new Picone-type identity for half-linear differential equations of second order and developed Sturmian comparison theory for both forced and unforced half-linear equations based on this identity [3, 5]. There are many studies dealing with Sturm comparison and oscillation results for a pair of elliptic type operators. We refer to Kreith [6, 7], Swanson [8] for Sturmian comparison theorems for linear elliptic equations and to Allegretto [9], Allegretto and Huang [10, 11], Bognar and Dosly [12],Dunninger [13], Kusano et al. [14], Yoshida [15, 16, 17] for Picone identities, Sturmian comparison and/or oscillation theorems for half-linear elliptic differential equations.

In this study we investigated one discontinuous eigenvalue problem which consists of Sturm-Liouville equation,

$$(pu')'(x) + q(x)u(x) = \lambda u(x) \tag{1.1}$$

to hold on two disjoint intervals $[-1, 0)$ and $(0, 1]$, where discontinuity in u and u' at the interior singular point $x = c$ are prescribed by transmission conditions

$$u(0-) = u(0+), \quad u'(0+) - u'(0-) = \delta u(0), \tag{1.2}$$

together with the boundary conditions

$$u(-1) = u(1) = 0 \tag{1.3}$$

where $p(x)$ and $p(x)$ are real valued, $p(x) > 0$, the potential $q(x)$ is continuous on $[-1, 0) \cup (0, 1]$ and has a finite limits $q(c\mp) = \lim_{x\to 0\mp} q(x)$; λ is a complex eigenparameter. Transmission problems appear frequently in various fields of physics and technics. For example, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see, [18] and the references listed therein). In recent years, Sturm-Liouville problems with transmission conditions have been an important research topic in mathematical physics [19, 20, 22, 23, 21, 24, 25, 26]. We give a method for proving the comparison and oscillation theorem of the discontinuous Sturm-liouville problem $(1.1) - (1.3)$.

Our aim in this paper is to establish comparison and oscillation results for discontinuous Sturm-Liouville problems with additional transmission conditions at the point of discontinuity.

2 Comparison and Oscillation Theorems

We will establish the Sturm comparison theorem for transmission problems in the following form.

Theorem 2.1. *Let $u = u_1(x)$ be a non-trivial solution of the equation*

$$(pu')'(x) + q(x)u(x) = \lambda_1 u(x), \quad x \in [-1, 0) \cup (0, 1] \tag{2.1}$$

satisfying transmission conditions at the point of interaction $x = 0$ given by

$$u(0-) = u(0+), \quad u'(0+) - u'(0-) = \delta u(0), \quad p(0-) = p(0+) \tag{2.2}$$

and $u = u_2(x)$ be a non-trivial solution of the equation

$$(pu')'(x) + q(x)u(x) = \lambda_2 u(x), \quad x \in [-1, 0) \cup (0, 1] \tag{2.3}$$

satisfying the same transmission conditions (2.2). If $\lambda_2 < \lambda_1$, then between any two consecutive zeros of $u_1(x)$ there is a zero of $u_2(x)$.

Proof. Let x_0, x_1 be consecutive zeros of $u_1(x)$ with $x_0 < x_1$. Suppose,that $u_2(x)$ does not have a zero on (x_0, x_1). Namely, suppose $u_1(x_1) = u_1(x_2) = 0$ and $u_2(x) \neq 0$ on (x_0, x_1). Let us consider equations of the form

$$L_i(u) = (pu')' + (q - \lambda_i)u = 0, \; i = 1,2 \quad x \in [-1,0) \cup (0,1] \tag{2.4}$$

By using the Lagrange's identity

$$u_2 L_1 u_1 - u_1 L_2 u_2 = \frac{d}{dx}\{p(x)(u_2 u_1' - u_1 u_2').\} \tag{2.5}$$

and the obvious equality

$$L_1(u_2) - L_2(u_2) = (\lambda_2 - \lambda_1)u_2 \tag{2.6}$$

we have

$$u_2 L_1 u_1 - u_1[L_2 u_2 + (\lambda_2 - \lambda_1)u_2] = (p(x)(u_2 u_1' - u_1 u_2'))'. \tag{2.7}$$

Then integrating on both sides of the equation (2.7) from x_0 to x_1, we get

$$[p(x)(u_1' u_2 - u_2' u_1)]|_{x_0}^{x_1} = \int_{x_0}^{x_1} (\lambda_2 - \lambda_1) u_1 u_2 dx > 0. \tag{2.8}$$

However, the left hand side reduces to

$$p(x_1)u_1'(x_1)u_2(x_1) - p(x_0)u_2(x_0)u_1'(x_0). \tag{2.9}$$

Therefore we find that

$$p(x_1)u_1'(x_1)u_2(x_1) - p(x_0)u_2(x_0)u_1'(x_0) = (\lambda_2 - \lambda_1) \int_{x_0}^{x_1} u_1(x)u_2(x)dx \tag{2.10}$$

Case 1. Let $(x_0, x_1) \subset [-1, 0)$.

i)) Let $u_1(x), u_2(x) > 0$ on (x_0, x_1). These conditions ensure that the integral on the right in (2.10) is positive. On the left, since $u_1(x) > 0$ by assumption, the function in increasing at the point x_0. Thus $u_1'(x_0) \geq 0$. but $u_1(x)$ cannot vanish at the point $x = x_1$ because then it would follow from the uniqueness theorem for the solutions of (2.1) that $u_1(x) \equiv 0$, which is impossible. So, $u_1'(x_0) > 0$. By similar method $u_1'(x_1) < 0$. Since $p(x) > 0$, $u_1(x_0), u_2(x_1) > 0$, $u_1'(x_0) > 0$ and $u_1'(x_1) < 0$ the left hand side of the equation 2.10 is nonpositive,but the right-hand side is positive. Hence we obtain a contradiction. Thus between any two consecutive zeros of $u_1(x)$ there is at least one zero of $u_2(x)$.

ii)Assume that both $u_1(x) < 0$ and $u_2(x) < 0$ are negative in the interval (x_0, x_1). This could be represented in a similar way to the case **i))**.

iii) Let $u_1(x) < 0$ and $u_2(x) > 0$ in the interval (x_0, x_1). These conditions ensure that the integral on the right in (2.10) is negative. However, on the left, we have $u_1(x_0) = u_1(x_1) = 0$ with $u_1'(x_0) < 0$ and $u_1'(x_1) > 0$. The left hand side therefore becomes

$$p(x_1)u_1'(x_1)u_2(x_1) - p(x_0)u_2(x_0)u_1'(x_0) > 0 \tag{2.11}$$

which presents us with a contradiction right-hand side negative and left-hand side positive. Hence we obtain a contradiction. Thus $u_2(x) = 0$ has at least one zero between the consecutive zeros x_0, x_1 of $u_1(x)$.

iv) Let $u_1(x) > 0$ and $u_2(x) < 0$ in the interval (x_0, x_1). This is similar to previous case.

Case 2. Let $(x_0, x_1) \subset (0, 1]$ This case easily proved similarly to case 1).

Case 3. Let $-1 \leq x_0 < 0 < x_1 \leq 1$. Integrating on both sides of the equation (2.7) over $[x_0, 0)$ to $(0, x_1]$ we get

$$[p(x)(u_1'u_2 - u_2'u_1)]|_{x_0}^{0-} + [p(x)(u_1'u_2 - u_2'u_1)]|_{0+}^{x_1} = (\lambda_2 - \lambda_1)[\int_{x_0}^{0-} u_1u_2dx + \int_{0+}^{x_1} u_1u_2dx]. \tag{2.12}$$

Since $u_1(x_0) = u_1(x_1) = 0$ we obtain

$$p(0-)(u_1'(0-)u_2(0-) - u_2'(0-)u_1(0-)) - p(0+)(u_1'(0+)u_2(0+) - u_2'(0+)u_1(0+))$$
$$+p(x_1)u_1'(x_1)u_2(x_1) - p(x_0)u_1'(x_0)u_2(x_0) = (\lambda_2 - \lambda_1)[\int_{x_0}^{0-} u_1(x)u_2(x)dx + \int_{0+}^{x_1} u_1(x)u_2(x)dx]. \tag{2.13}$$

Using the transmission conditions

$$u(0-) = u(0+), \quad u'(0+) - u'(0-) = \delta u(0), \quad p(0-) = p(0+)$$

we find

$$p(0-)(u_1'(0-)u_2(0-) - u_2'(0-)u_1(0-)) = p(0+)(u_1'(0+)u_2(0+) - u_2'(0+)u_1(0+)).$$

By writing this result in (2.13) we get

$$p(x_1)u_1'(x_1)u_2(x_1) - p(x_0)u_1'(x_0)u_2(x_0) = (\lambda_2 - \lambda_1)[\int_{x_0}^{0-} u_1(x)u_2(x)dx + \int_{0+}^{x_1} u_1(x)u_2(x)dx].$$

Without loss of generality we shall assume that $u_1(x) > 0$ and $u_2(x) > 0$ in the interval $[x_0, 0) \cup (0, x_1)$. Thus, the right-hand side of the last equality is positive. On the left, since $u_1(x) > 0$ by assumption, the function in increasing at the point x_0. In this case $u_1'(x_0) > 0$. By the same method $u_1'(x_1) < 0$. Consequently, he left hand side of the last equation is negative. This is a contradiction. The proof is complete.

□

Theorem 2.2. *There are an infinitely increasing sequence of eigenvalues* $\lambda_0, \lambda_1, \lambda_2, ...$ *of the boundary value problem* $(1.1)-(1.3)$. *Moreover, if* $W(\varphi(x, \lambda_1), \varphi(x, \lambda_2); 0+) < 0$ *than the eigenfunction corresponding to the eigenvalue* λ_n *has exactly* n *zeros on* $[-1, 0) \cup (0, 1]$.

References

[1] C. Sturm, Mmoire sur les quations differentielles linaires du second ordre. J. Math. Pures Appl. 1, 106-186 (1836)

[2] M. Picone, Sui valori eccezionali di un parametro da cui dipende unequazione differenziale lineare ordinaria del secondordine. Ann. Sc. Norm. Super. Pisa 11, 1-141 (1909)

[3] T. Kusano, J. Jaro and N. Yoshida, Picone-type inequalities for half-linear elliptic equations and their applications. Adv. Math. Sci. Appl. 12(2), 709-724 (2002)

[4] J. Jaro, T. Kusano, Second-order semilinear differential equations with external forcing terms. RIMS Kokyuroku 984, 191-197 (1997)

[5] J. Jaro, T. Kusano, A Picone type identity for second order half-linear differential equations. Acta Math. Univ. Comen. 68, 117-121 (1999)

[6] K. Kreith, Oscillation Theory. Lecture Notes in Mathematics, vol. 324. Springer, Berlin (1973)

[7] K. Kreith, Picones identity and generalizations. Rend. Mat.8, 251-261 (1975)

[8] CA. Swanson, Comparison and Oscillation Theory of Linear Differential Equations. Academic Press, New York (1968)

[9] W. Allegretto, Sturm theorems for degenerate elliptic equations. Proc. Am. Math. Soc. 129, 3031-3035 (2001).

[10] W. Allegretto, YX. Huang, A Picones identity for the p-Laplacian and applications. Nonlinear Anal. 32, 819-830 (1998)

[11] W. Allegretto, YX. Huang, Principal eigenvalues and Sturm comparison via Picones identity. J. Differ. Equ. 156, 427-438 (1999).

[12] G. Bognr, O. Dosly, The application of Picone-type identity for some nonlinear elliptic differential equations. Acta Math. Univ. Comen. 72, 45-57 (2003).

[13] DR. Dunninger, A Sturm comparison theorem for some degenerate quasilinear elliptic operators. Boll. Unione Mat. Ital., A (7) 9, 117-121 (1995)

[14] T. Kusano, J. Jaro, N. Yoshida, A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order. Nonlinear Anal. 40, 381-395 (2000)

[15] N. Yoshida, Oscillation criteria for half-linear partial differential equations via Picones identity. In: Proceedings of Equadiff, vol. 11, pp. 589-598 (2005)

[16] N. Yoshida, A Picone identity for half-linear elliptic equations and its applications to oscillation theory. Nonlinear Anal. TMA 71, 4935-4951 (2008)

[17] N. Yoshida, Oscillation Theory of Partial Differential Equations. World Scientific, Singapore (2008).

[18] H. Pham Huy and E. Sanchez-Palencia, *Phenom'enes des transmission 'a travers des couches minces de conductivite elevee* J. Math. Anal. Appl., 47 (1974), pp. 284-309.

[19] B. P. Allahverdiev, E. Bairamov and E. Ugurlu, *Eigenparameter dependent Sturm-Liouville problems in boundary conditions with transmission conditions*, Journal of Mathematical Analysis and Applications, 401, No. 1, 388396 (2013).

[20] I. Akbarfam, A. Jodayree, *Resolvent Operator and Self-Adjointness of Sturm-Liouville Operators with a Finite Number of Transmission Conditions*, Mediterranean Journal Of Mathematics, 11(2), 447-462(2014).

[21] J. Ao and J. Sun *Matrix representations of SturmLiouville problems with coupled eigenparameter-dependent boundary conditions*, Applied Mathematics and Computation 244 (2014) 142148

[22] K. Aydemir and O. Sh. Mukhtarov,*Completeness Of One Two-Interval Boundary Value Problem With Transmission Conditions*, Miskolc Mathematical Notes, 15(2), 293-303(2014).

[23] E. Bairamov and E. Ugurlu, *On the characteristic values of the real component of a dissipative boundary value transmission problem,* Appl. Math. and Comp. 218(2012), 9657-9663.

[24] F. Hıra and N. Altınışık, *Sampling theorems for Sturm-Liouville problem with moving discontinuity points*, Boundary Value Problems 2014, 2014:237.

[25] M. Kadakal and O. Sh. Mukhtarov *SturmLiouville problems with discontinuities at two points*, Computers and Mathematics with Applications 54(2007) 1367-1379

[26] O. Sh. Mukhtarov, H.Olğar, K. Aydemir *Transmission problems for the Sturm-Liouville equation involving an abstract linear operator* IP Conference Proceedings; 8/20/2014, Vol. 1611, 325-332.